Modified Kuramoto-Sivashinsky equation: Stability of stationary solutions and the consequent dynamics

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We study the effect of a higher-order nonlinearity in the standard Kuramoto-Sivashinsky equation: $\partial_x \widetilde{G}(H_x)$. We find that the stability of steady states depends on dv/dq, the derivative of the interface velocity on the wave vector q of the steady state. If the standard nonlinearity vanishes, coarsening is possible, in principle, only if $\widetilde{G}$ is an odd function of H_x. In this case, the equation falls in the category of the generalized Cahn-Hilliard equation, whose dynamical behavior was recently studied by the same authors. Alternatively, if $\widetilde{G}$ is an even function of H_x, we show that steady-state solutions are not permissible.

I. INTRODUCTION

One of the most prominent and generic equations that arises in nonequilibrium systems is the Kuramoto-Sivashinsky (KS) [1–3] equation

$$H_t + H_{xxxx} + H_{xx} + HH_x = 0, \tag{1}$$

where H is some scalar function (like the slope of a one-dimensional growing front) and differentiations are subscripted. The linear stability analysis of the KS equation (by looking for solutions in the form of $e^{iqx+\omega t}$) yields $\omega = q^2 - q^4$. The (linearly) fastest-growing mode has a wave number given by $q_u = 1/\sqrt{2}$ (obtained from $\partial_q \omega = 0$). For a large box size the KS equation is known to exhibit spatiotemporal chaos. The chaotic pattern statistically selects a length scale which is close to $2\pi/q_u$: in fact, the structure factor $\langle|\widetilde{H}_q|^2\rangle$, where $\langle\cdots\rangle$ designates the average over many runs and $\widetilde{H}_q$ is the Fourier transform of H, exhibits a maximum around $q = q_u$. Other nonequilibrium equations are known, however, to exhibit different dynamical behaviors: just to limit ourselves to one-dimensional systems, we may have coarsening, a diverging amplitude with a fixed wavelength, a frozen pattern, traveling waves, and so on.

An important issue is the recognition of general criteria that enable one to predict whether or not coarsening takes place within a class of nonlinear equations, without having to resort to a forward time-dependent calculation. In recent works [4,5] we have considered several classes of one-dimensional partial differential equations (PDE's), having the form $H_t = \mathcal{N}[H]$, where $\mathcal{N}$ is a nonlinear operator acting on the spatial variable x.

Sometimes, even in the presence of strong nonlinearities, the search for steady states reduces to solving a Newton-type equation $H_{xx}^* + V(\widetilde{H}^*) = 0$, where H^* is some function of H. In these cases, $\lambda(A)$, giving the dependence of the wavelength λ of the steady state on its amplitude A, is a one-value function

(see Fig. 1, solid lines), and the criterion for the existence of coarsening is expressed in terms of the derivative $\lambda'(A)$.

It has been shown that the sign of $-\lambda'(A)$ is the same as that of the phase diffusion coefficient. Thus $\lambda'(A) > 0$ corresponds to a branch which is unstable with respect to the phase of the pattern, entailing thus coarsening. The situation is more complicated when $\lambda(A)$ exhibits a fold (see Fig. 1, dashed line). This event occurs, e.g., in the KS equation and in the Swift-Hohenberg equation. As for the KS equation, which is the topic of this paper, Nepomnyashchii [1] has shown that the forearm part of the curve $\lambda(A)$ with positive slope, $\lambda'(A) > 0$, is an unstable branch. This result holds for the pure KS equation, however [6].

The aim of this paper is the following. (i) First, we shall extend the result of Nepomnyashchii [1] to a generalized form of the KS equation, which includes higher-order nonlinearities. The KS equation has been derived in a number of physical problems, including hydrodynamics, crystal growth, chemical waves, etc. (for a review, see Ref. [7]). The KS equation is valid to leading order in the expansion. Next nonlinear terms may affect dynamics. Indeed, the next lead-
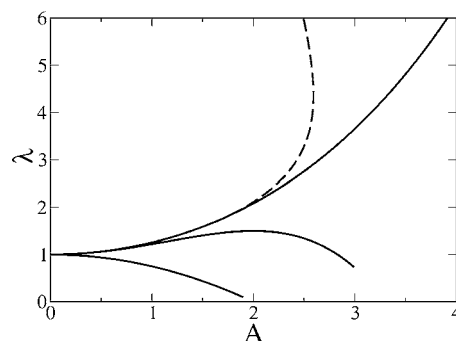

FIG. 1. The periodic stationary solutions of some classes of nonlinear equations that satisfy a Newton-type equation $H_{xx}^* + V(H^*) = 0$, and the resulting $\lambda(A)$ is a single-value function. According to the explicit form of $V(H^*)$, we may have the different curves shown as solid lines. The dashed line, which displays a fold, corresponds to a non-single-value function. It comes out in the Kuramoto-Sivashinsky and Swift-Hohenberg equations. Units on both axes are arbitrary.

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ing term in the KS equation (H_xH_{xx}) has been analyzed in [8], and it has been shown that this term significantly affects the dynamics; for example, the profile may exhibit deep grooves. We shall consider a more general form of the modified KS equation by adding a term like $\partial_x \widetilde{G}(H_x)$ (this includes as a particular case the term H_xH_{xx}). We find that the stability of the steady-state solutions depends on $v'(q)$, where v has, in a front problem, the meaning of the average interface velocity. (ii) If the standard KS nonlinearity vanishes (no HH_x term), then there is coarsening if $\widetilde{G}$ is an odd function. In this case a mapping of the equation onto a generalized Cahn-Hilliard equation is straightforward. (iii) If $\widetilde{G}$ is even (still in the absence of the standard nonlinearity), we show that there exists no steady-state periodic solution, as attested to by numerical simulations for $\widetilde{G}=H_x^2$ [9].

II. MODIFIED KS EQUATION

A. The method

We study the following equation:

$$H_t + c_4H_{xxxx} + c_2H_{xx} + \alpha HH_x + \beta\partial_x\widetilde{G}(H_x) = 0, \quad (2)$$

which reduces to the standard Kuramoto-Sivashinsky equation when $\widetilde{G}=0$. A rescaling of x, t, and H always allows one to reduce the equation to a one-parameter equation, which can be absorbed into a redefinition of $\widetilde{G}$. However, for the sake of clarity, we do not get rid of α, so that we write

$$H_t + H_{xxxx} + H_{xx} + \alpha HH_x + \partial_x\widetilde{G}(H_x) = 0. \quad (3)$$

It is also useful to rewrite Eq. (3) using the variable u, with $u_x=H$:

$$u_t + u_{xxxx} + u_{xx} + \frac{\alpha}{2}u_x^2 + \widetilde{G}(u_{xx}) = 0, \quad (4)$$

where the integration constant v_0 can be canceled out by the transformation $u(x,t) \rightarrow u(x,t) - v_0t$.

Within the H formulation, the average $d\langle H\rangle/dt$ vanishes, because Eq. (3) has the conserved form $H_t=-\partial_x(\cdots)$. Within the u formulation,

$$\frac{d\langle u\rangle}{dt} = -\left[\frac{\alpha}{2}\langle u_x^2\rangle + \langle\widetilde{G}(u_{xx})\rangle\right] = v. \quad (5)$$

For interfacial problems u is the interface profile, and therefore v represents the average speed of the front. For steady-state solutions v is related nonlinearly to the front profile $u_0(x)$; see Eq. (5).

We start from a stationary solution of period q, $H(x)$, and perturb it by adding $h(x)\exp(-\omega t)$. The function $h(x)$ therefore satisfies the linear equation

$$h_{xxxx} + h_{xx} + \alpha(hH)_x + (h_xG(H_x))_x = \omega h, \quad (6)$$

where $G=\widetilde{G}'$ and whose coefficients are periodic with period $\lambda=2\pi/q$. According to the Floquet-Bloch theorem, the solution has the form $h(x)=\exp(iKx)F(x)$, where $F(x)$ has the same period as $H(x)$.

We are interested in weak modulations of long wavelength—i.e., with $K \ll q$. It is therefore convenient to introduce the reduced wave vector $Q=K/q$ and the phase $\phi=qx$. The equation determining the steady state $H(\phi)$ is

$$q^4H_{\phi\phi\phi\phi} + q^2H_{\phi\phi} + \alpha qHH_\phi + q^2H_{\phi\phi}G(qH_\phi) = 0, \quad (7)$$

and Eq. (6) reads

$$q^4h_{\phi\phi\phi\phi} + q^2h_{\phi\phi} + q\alpha(hH)_\phi + q^2(h_\phi G(qH_\phi))_\phi \equiv \mathcal{L}h = \omega h, \quad (8)$$

with $h(\phi)=\exp(iQ\phi)F(\phi)$.

The final step is to expand both ω and $F(\phi)$ in powers of Q,

$$F(\phi) = F_0(\phi) + QF_1(\phi) + Q^2F_2(\phi) + \cdots, \quad (9)$$

$$\omega = \omega_0 + Q\omega_1 + Q^2\omega_2 + \cdots, \quad (10)$$

and to solve Eq. (8) at the lowest orders in Q.

B. Zeroth order

The differential equation determining the steady state $H(x)$ does not depend explicitly on x, so that $H(x+x_0)$ is a solution as well. This symmetry implies that $\mathcal{L}h=\omega h$ [see Eq. (8)] is solved by $h=H_\phi$ and $\omega=0$, as can be easily checked by taking the ϕ derivative of Eq. (7). Since the zeroth-order equation is simply

$$\mathcal{L}F_0 = 0, \quad (11)$$

we obtain $F_0=H_\phi$.

C. First order

The equation for F_1 reads

$$\mathcal{L}F_1 = \omega_1H' - iq[4q^3H'''' + 2q(1+G)H'' + (\alpha H + q^2H''G')H'], \quad (12)$$

where we have used the shorthand notation $H'=H_\phi$, $H''=H_{\phi\phi},\ldots$. If we differentiate Eq. (7) with respect to q, we get a similar equation,

$$\mathcal{L}H_q = -(4q^3H'''' + 2qH'' + \alpha HH' + 3\beta q^2H'H''), \quad (13)$$

where H_q is the derivative of H with respect to q.

The comparison of Eqs. (12) and (13) suggests one to look for F_1 in the form $F_1=iqH_q+c$, where c is a constant. We easily find $c=\omega_1/(\alpha q)$, so that

$$F_1 = \frac{\omega_1}{\alpha q} + iqH_q. \quad (14)$$

This result shows that in the absence of the standard nonlinearity, $\alpha=0$, ω_1 should vanish whatever $\widetilde{G}$ is.

We might have started with an even more general Equation (3), replacing the standard nonlinearity (αHH_x) $=\partial_x(\frac{\alpha}{2}H^2)$ with $\partial_x\widetilde{P}(H)$. The term $\alpha HH'$ on the right-hand sides of Eqs. (12) and (13) would be replaced by $P(H)H'$, with $P=\widetilde{P}'$. However, in the general case P' is not a con-

stant: therefore, it is not possible to look for a solution $F_1 = iqH_q + c$.

D. Second order

The equation for F_2 has the form

$$\mathcal{L}F_2 + q^2(1 + G)(-F_0 + 2iF_1') + (\alpha qH + q^3H''G')iF_1$$
$$= \omega_2F_0 + \omega_1F_1 \qquad (15)$$

or

$$\mathcal{L}F_2 = \omega_2H' + iq\omega_1H_q + q^2H' + 2q^3H_q' - i\omega_1H$$
$$- i(\omega_1/\alpha)q^2H''G'(qH') + \omega_1^2/(\alpha q) + q^2G(qH')H'$$
$$+ 2q^3G(qH')H_q' + \alpha q^2HH_q + q^4H''G'(qH')H_q. \qquad (16)$$

Now we take the 2π average of the previous equation, getting

$$\frac{\omega_1^2}{\alpha q} + q^2\langle G(qH')H'\rangle + 2q^3\langle G(q(H')H_q'\rangle + \alpha q^2\langle HH_q\rangle$$
$$+ q^4\langle H''G'(qH')H_q\rangle = 0. \qquad (17)$$

Finally, we obtain

$$\omega_1^2 = -\alpha q^3\frac{d}{dq}\left[\frac{\alpha}{2}\langle H^2\rangle + \langle\widetilde{G}(qH')\rangle\right] \equiv \alpha q^3\frac{d}{dq}v. \qquad (18)$$

This result proves that $\omega_1 = 0$ if $\alpha = 0$, whatever the function $\widetilde{G}$ is. Consider the case $\alpha \neq 0$ (we are at liberty to choose $\alpha > 0$). Since $\omega < 0$ signals an instability, one sees that if $\frac{d}{dq}v > 0$, the periodic solution is unstable, because there is a real solution $\omega_1 < 0$. This generalizes the result of [1], obtained for the pure KS equation, to the higher-order KS equation. It is only in the pure KS limit that the spectrum of stability is related to the slope of the steady amplitude. In the higher-order equation, however, this ceases to be the case. Instead we should replace the amplitude by the drift velocity, a quantity which can be still obtained from pure steady-state considerations. It is to be noted that the criterion regarding the slope of v, $\frac{d}{dq}v > 0$, is a sufficient condition for instability. In the opposite limit no conclusion can be drawn, since there is a need to push the calculation to the next order. This has not been checked yet, but it is likely that the analysis is too much involved in order to lend itself to analytical tractability.

E. Determination of ω_2

The determination of ω_2 implies the resolution of the differential equation

$$\mathcal{L}^\dagger u = 0, \qquad (19)$$

where

$$\mathcal{L}u = u_{xxxx} + u_{xx} + \alpha(Hu)_x + [G(H')u_x]_x \qquad (20)$$

and

$$\mathcal{L}^\dagger u = u_{xxxx} + u_{xx} - \alpha Hu_x + [G(H')u_x]_x. \qquad (21)$$

The equation $\mathcal{L}u = 0$ is solved by $u = H'$, but we do not know the solution of $\mathcal{L}^\dagger u = 0$. $\mathcal{L}^\dagger \neq \mathcal{L}$ because of the α term. If $\alpha = 0$, such a term is absent and $\mathcal{L}^\dagger = \mathcal{L}$. This case is treated in the next subsection.

F. The case $\alpha = 0$

Stationary solutions are determined [see Eq. (3)] from the condition

$$\partial_x\left[H_{xxx} + H_x + \frac{\alpha}{2}H^2 + \widetilde{G}(H_x)\right] = 0, \qquad (22)$$

which can be integrated once, giving

$$H_{xxx} + H_x + \frac{\alpha}{2}H^2 + \widetilde{G}(H_x) = C, \qquad (23)$$

where C is a constant.

If $\alpha = 0$, we can set $h = H_x$ and stationary solutions are given by the equation

$$h_{xx} = -h - \widetilde{G}(h) + C. \qquad (24)$$

The equation for $H(x)$ admits periodic solutions only if $h(x)$ itself is periodic and has zero average for any initial condition (such that the solution is bounded). This requires that $C = 0$ and $\widetilde{G}(h)$ be an odd function.

In the same limit $\alpha = 0$, the full PDE (3) is written

$$H_t = -\partial_x[H_{xxx} + H_x + \widetilde{G}(H_x)], \qquad (25)$$

and taking the spatial derivative of both terms, we get

$$h_t = -\partial_{xx}[h_{xx} + h + \widetilde{G}(h)], \qquad (26)$$

where $h = H_x$, as before. We have therefore obtained a generalized Cahn-Hilliard equation, whose dynamical behavior is known to show coarsening if and only if the wavelength λ of steady states is an increasing function of their amplitude A [5]. We have reobtained the same result following the method discussed in this section (calculations are not shown).

III. FINAL REMARKS

The present and recent work [4,5] has the main objective to find general criteria to understand and anticipate the dynamics of nonlinear systems by the analysis of steady-state solutions only. For some important classes of PDE's, which have the common feature of a single-value $\lambda(A)$ function, the criterion is based on the sign of the derivative $\lambda'(A)$. In this Brief Report we have considered a modified, generalized Kuramoto-Sivashisnky equation, where the $\lambda(A)$ curve is not single value, because it displays a fold. We have therefore established a different criterion, based on dv/dq, the derivative of the interface velocity on the wave vector q of the steady state.

If the standard KS nonlinearity is absent, we can say more. If the nonlinearity $\partial_x \widetilde{G}(H_x)$ corresponds to an even function $\widetilde{G}$, the equation does not support periodic stationary solutions and this prevents coarsening in principle: we have a pattern of fixed wavelength and diverging amplitude. Instead, if $\widetilde{G}$ is an odd function, the equation falls into a previously studied class, the generalized Cahn-Hilliard equation, which can show different behaviors according to the form of $\lambda(A)$. Finally, it is an important task for future investigations to see whether or not information of the types presented here and in [4,5] has analogs in higher dimensions.

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